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Generalised hyperbolic functions and some physical applications

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Abstract. We consider the so-called generalised hyperbolic functions and their properties. In particular, we find some new relationships with the Bessel functions and that they occur naturally in many physical systems whenever a Z_N symmetry appears. Examples are Z_N-symmetric spin systems as well as electronic systems with periodic boundary conditions.

1. Introduction

Several problems in theoretical physics display cyclic permutational symmetry which could be conveniently described as Z_N symmetry, where Z_N is the usual discrete Abelian group of order N. The simplest example is the one-body problem in one dimension with periodic boundary conditions. The wavefunction is naturally a plane wave with a momentum k obeying the quantisation condition $(\exp \mathrm{i}k)^N = 1$ where N is the number of sites on the lattice. The involutional property of $\exp \mathrm{i}k$ can be carried over to matrices, then we have $\mathbf{A}^N = \mathbf{I}$ in some systems involving spins which can take values on the N regular positions of the unit circle (e.g. the so-called 'clock' model). Through the process of exponentiation, which is motivated basically by the Boltzmann weights in statistical mechanics, numbers and matrices which are involutory are directly related to the generalised hyperbolic functions.

These functions made their first appearance in the work of Poli (Poli 1946) and are to be understood as generalisations of the usual hyperbolic functions. So instead of a basis consisting of $\cosh x$ and $\sinh x$ we shall have N functions playing the same rôle. They also obey a simple differential equation with constant coefficients, but of order N. As we shall see, they have simple addition rules and are handy to treat some physical problems. Moreover, we shall show that they are connected to Bessel functions via summation formulae. Thus they do not occur as a mathematical curiosity but present another aspect of the Bessel functions.

This paper is organised as follows. In section 2 we shall introduce the generalised hyperbolic functions and the various definitions which concern them. We then present their properties: derivative, integral, addition rules and asymptotic behaviour. Section 3 is devoted to the relation with Bessel functions and some additional properties. In section 4 we present some simple applications in statistical mechanics: the one-dimensional Potts models. Our aim is to demonstrate the usefulness of these functions as they have already appeared in our recent work on the atomic limit of the one-dimensional Hubbard model (Audit and Truong, 1989).

2. Generalised hyperbolic functions of order N

To simplify the approach we define the generalised hyperbolic functions as entire functions given by the power series:

$$\lambda_j^N(x) = \sum_{k=0}^{\infty} \frac{x^{j+kN}}{(j+kN)!} \qquad j = 0, 1, 2, \ldots, N-1. \tag{1}$$

We observe that this is a piece of the exponential series. Hence each of the $\lambda_j^N(x)$ is bounded by $\exp|x|$. For $N=2$ we recover immediately

$$\lambda_0^2(x) = \cosh x \qquad \lambda_1^2(x) = \sinh x. \tag{2}$$

Since these two hyperbolic functions are simply linear combinations of exponentials, it is also natural to look for such representation for the generalised hyperbolic functions of order N. We have:

$$\lambda_j^N(x) = \frac{1}{N} \sum_{k=0}^{N-1} \omega^{-jk} \exp(\omega^k x) \tag{3}$$

where ω is the Nth primitive root of unity (i.e. $\omega^N = 1$). We observe that always $\lambda_j^N(0) = \delta_{j,0}$. This is due to the fact that $1+\omega+\ldots+\omega^{N-1} = 0$. So we may say that we have for general N one cosh x-like function: the λ_0^N, and $(N-1)$ sinh x-like functions although parity is not an obvious property.

Inverting (3) yields N generating functions for the generalised hyperbolic functions:

$$\exp(\omega^p x) = \sum_{j=0}^{N-1} \omega^{pj} \lambda_j^N(x). \tag{4}$$

It is clear that all the $\lambda_j^N(x)$ are real functions because the N roots ω^j, $j = 0, 1, \ldots, (N-1)$ are distributed symmetrically with respect to the real axis in the complex plane. If N is even then there is an extra symmetry in the sense that the terms in (3) may be pairwise grouped so that the λ_j^N are given by

$$\lambda_j^{2p} = \frac{1}{2p} \{[\exp(x) + (-1)^j \exp(-x)] + \omega^{-j}[\exp(\omega x) + (-1)^{-j} \exp(-\omega x)] + \ldots$$

$$+ \omega^{-j(p-1)}[\exp(\omega^{p-1})x + (-1)^j \exp(-\omega^{p-1}x)]\}. \tag{5}$$

Since $\omega^p = -1$, depending on the parity of j, we will get a sum of cosh or sinh.

The $\lambda_j^N(x)$ are also periodic with respect to the index j:

$$\lambda_j^N(x) = \lambda_{j+N}^N(x). \tag{6}$$

As it is known from the usual hyperbolic functions, derivation and integration are equivalent to shifting the index forward and backward:

$$\frac{\mathrm{d}}{\mathrm{d}x} \lambda_j^N(x) = \lambda_{j-1}^N(x) \qquad \int \lambda_j^N(x)\,\mathrm{d}x = \lambda_{j+1}^N(x). \tag{7}$$

This gives the results that the function $\lambda_0^N(x)$ is $(N-1)$ flat at the origin, $\lambda_1^N(x)$ has a slope 1 and an inflexion point at the origin, $\lambda_2^N(x)$ is only simply flat and has a non-zero curvature at the origin, etc.

Asymptotically we see that for large $x \to \infty$, since $\lambda_j^N(x)$ consists of exponentials (see (3)), it will be dominated by the largest exponential:

$$\lambda_j^N(x) \sim \frac{1}{N} \exp(x). \tag{8}$$

Thus all the λ curves are bounded by $\exp(x)$.

To obtain the addition rules we compute the following identity in two different ways: $\exp[(x+y)\omega^p] = \exp(x\omega^p)\exp(y\omega^p)$ using the generating function in (4). The result is simply:

$$\lambda_l^N(x+y) = \sum_{j+k=l} \lambda_j^N(x)\lambda_k^N(y) \tag{9}$$

it generalises the known identities for cosh and sinh. In particular, we have taken $l=0$ and $x=-y$ in the previous formula

$$\lambda_0^N(x)\lambda_0^N(-x) + \lambda_1^N(x)\lambda_{N-1}^N(-x) + \ldots + \lambda_{N-1}^N(x)\lambda_1^N(-x) = 1 \tag{10}$$

generalising the identity: $\cosh^2(x) - \sinh^2(x) = 1$. Finally iterating the differentiation of (7) and using the periodicity condition of (6), we arrive at the differential equation for the $\lambda_j^N(x)$ functions:

$$\frac{\mathrm{d}^N}{\mathrm{d}x^N}\lambda_j^N(x) = \lambda_j^N(x). \tag{11}$$

There also exists a 'trigonometric' version of the $\lambda_j^N(x)$ functions (Erdelyi *et al* 1955) but we shall not dwell on them here.

3. Relation to the modified Bessel functions

This connection seems to be new and it is based on the generating function of the modified Bessel functions (Magnus *et al* 1966):

$$\exp(z\cos\theta) = I_0(z) + 2\sum_{k=1}^{\infty} I_k(z)\cos(k\theta). \tag{12}$$

Putting $\theta = p2\pi/N$ for $p = 0, 1, \ldots, (N-1)$, and using the fact that

$$\exp\left(z\cos p\frac{2\pi}{N}\right) = \exp\left(\frac{z}{2}\exp(\mathrm{i}p2\pi/N)\right)\exp\left(\frac{z}{2}\exp(-\mathrm{i}p2\pi/N)\right)$$

we can insert the generating function in (4) to obtain the basic identity:

$$\exp\left(z\cos p\frac{2\pi}{N}\right) = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\lambda_m^N\left(\frac{z}{2}\right)\lambda_n^N\left(\frac{z}{2}\right)\omega^{p(m-n)} \tag{13}$$

where $\omega = \exp(2\mathrm{i}\pi/N)$. The infinite sum on the Bessel functions may be thus replaced advantageously by the double finite sums on the λ functions:

$$I_0(z) + 2\sum_{k=1}^{\infty} I_k(z)\cos kp\frac{2\pi}{N} = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\lambda_m^N\left(\frac{z}{2}\right)\lambda_n^N\left(\frac{z}{2}\right)\omega^{p(m-n)} \tag{14}$$

for $p = 0, 1, \ldots, (N-1)$.

Now using the fact that

$$\sum_{p=0}^{N-1} 2\cos kp\frac{2\pi}{N} = 2N\delta_{k,qN}$$

where q is an integer, we can sum in (14) over p from 0 to $(N-1)$ to obtain a new identity for the generalised hyperbolic functions:

$$I_0(z)+2\sum_{h=1}^{\infty} I_{hN}(z)=\sum_{n=0}^{N-1}\left(\lambda_n^N\left(\frac{z}{2}\right)\right)^2. \tag{15}$$

In fact, (14) really represents N different identities for each p from 0 to $(N-1)$. This system can be simplified as follows:

$$\sum_{l=0}^{N-1} \omega^{lp} C_l(z)=I_0(z)+2\sum_{k=1}^{\infty} I_k(z)\cos kp\frac{2\pi}{N} \tag{16}$$

where

$$C_l(z)=\sum_{k=0}^{N-l-1} \lambda_k^N\left(\frac{z}{2}\right)\lambda_{k+l}^N\left(\frac{z}{2}\right)+\sum_{k=N-l}^{N-1} \lambda_k^N\left(\frac{z}{2}\right)\lambda_{k-l}^N\left(\frac{z}{2}\right)$$

for $p=0,1,\ldots,(N-1)$.

Generalised hyperbolic functions turn out to be useful in transforming some identity involving Bessel functions (Hansen 1975, Audit and Truong 1989). For example, we have

$$\sum_{k=0}^{\infty} c^k I_{a+k}(z)=z^{-a}\exp\left(\frac{cz}{2}\right)\int_0^z t^a\exp\left(-\frac{ct^2}{2z}\right)I_{a-1}(t)\,\mathrm{d}t \tag{17}$$

where a and c are two arbitrary numbers. We may choose $c=\exp(p2\mathrm{i}\pi/N)$ therein and sum over p. Then using the definition of the function $\lambda_0^N(x)$, we obtain the following remarkable identity:

$$\sum_{h=0}^{\infty} I_{a+hN}(z)=z^{-a}\int_0^z t^a I_{a-1}(t)\lambda_0^N\left(\frac{z}{2}-\frac{t^2}{2z}\right)\mathrm{d}t. \tag{18}$$

This has been used recently in the one-dimensional Hubbard model in the so-called atomic limit where we have exploited fully the properties of the $\lambda_0^N(x)$ function to compute the free energy with one hole present.

4. Some physical applications

We shall discuss here mainly applications in statistical mechanics, which extend a recent example in solid state physics (Audit and Truong 1989). Although some connections have been established before between two-dimensional Potts models and generalised hyperbolic functions (Kwasniewski 1985) we shall discuss mainly so-called one-dimensional Potts models.

Consider a chain of M sites with periodic boundary conditions. Each site carries a spin S_i which takes the values ω^p where $\omega=\exp(2\mathrm{i}\pi/N)$ and $p=0,1,\ldots,(N-1)$. The spins interact through only their nearest neighbours and the corresponding Boltzmann weight is given by

$$\exp\left(K\sum_{l=1}^{N-1} S_i^l S_{i+1}^{N-l}\right)$$

K being a coupling constant. Basically it means that the interaction energy (divided by kT) is equal to $(N-1)K$ if the neighbouring spins are lined up, if not then it is equal to $(-K)$. The partition function of the whole chain is equal to the sum on all possible configurations of the spins of the corresponding Boltzmann weights. This sum is evaluated normally with the use of a transfer matrix $\mathbf{T}$, namely:

$$\mathscr{Z}_M = \mathrm{Trace}(\mathbf{T})^M \tag{19}$$

whereby the matrix $\mathbf{T}$ is:

$$\begin{pmatrix} \exp[(N-1)K] & \exp(-K) & \dots & \exp(-K) \\ \exp(-K) & \exp[(N-1)K] & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \exp(-K) & \cdot & \dots & \exp[(N-1)K] \end{pmatrix}. \tag{20}$$

The problem of diagonalising the matrix $\mathbf{T}$ is now reduced to the resolution of an algebraic equation of order N, for which the explicit solution is quite difficult to obtain.

However, in the following we can show that a closed form of the trace of $\mathbf{T}^M$ can be derived using the generalised hyperbolic functions. First we observe that $\mathbf{T}$ can be written as:

$$\mathbf{T} = \exp[(N-1)K]\mathbf{I} + \exp(-K)(\mathbf{M} + \mathbf{M}^2 + \dots + \mathbf{M}^{N-1}) \tag{21}$$

where $\mathbf{M}$ is the cyclic matrix of order N given as:

$$\begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & 0 & \dots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 1 & \cdot & \dots & 0 \end{pmatrix}. \tag{22}$$

Clearly we have $\mathbf{M}^N = \mathbf{I}$. Now the trick is to calculate the trace of the operator $\exp(\beta\mathbf{T})$ instead of the trace of $\mathbf{T}^M$. Here β is the inverse temperature playing the role of a parameter. Now,

$$\exp(\beta\mathbf{T}) = \exp[\beta \exp(N-1)K] \exp(\beta\, \mathrm{e}^{-K})\mathbf{M} \dots \exp(\beta\, \mathrm{e}^{-K})\mathbf{M}^{N-1}.$$

Each term of the product may be expanded as in (4):

$$\exp(\beta\, \mathrm{e}^{-K})\mathbf{M}^p = \sum_{j=0}^{N-1} \lambda_j^N(\beta\, \mathrm{e}^{-K})\mathbf{M}^{pj}$$

hence after substitution one finds:

$$\exp(\beta\mathbf{T}) = \exp\left(\beta\, \mathrm{e}^{(N-1)K} \prod_{p=1}^{N-1} \sum_{j_p=0}^{N-1} \lambda_{j_p}^N(\beta\, \mathrm{e}^{-K})\mathbf{M}^{pj_p}\right). \tag{23}$$

Upon taking the trace on both sides of this equation we have:

$$\mathrm{Trace}(\exp \beta\mathbf{T}) = \exp\left(\beta\, \mathrm{e}^{(N-1)K} N \sum_{j_{p_1},\dots,j_{p_{N-1}}} \lambda_{j_{p_1}}^N(\beta\, \mathrm{e}^{-K}) \dots \lambda_{j_{p_{N-1}}}^N(\beta\, \mathrm{e}^{-K})\right) \tag{24}$$

with the condition

$$\sum_{k=1}^{(N-1)} k j_{p_k} = 0 \bmod N.$$

So finally the partition function $\mathscr{Z}_M$ is obtained as

$$\mathscr{Z}_M = \left(\frac{\mathrm{d}}{\mathrm{d}\beta}\right)^M \mathrm{Trace}(\exp \beta\mathbf{T})\big|_{\beta=0}. \tag{25}$$

Examples.

For $N=3$, the calculation is quite simple and yields:

$$\left(\frac{\mathrm{d}}{\mathrm{d}\beta}\right)^M [\mathrm{Trace}(\exp\beta\mathbf{T})]$$
$$=\exp\beta\, \mathrm{e}^{2K}[(\lambda_0^3)^2+(\lambda_1^3)^2+(\lambda_2^3)^2][(\mathrm{e}^{2K}+2\,\mathrm{e}^{-K})^M+2(\mathrm{e}^{2K}-\mathrm{e}^{-K})^M]$$
$$+\exp\beta\, \mathrm{e}^{2K}2[\lambda_0^3\lambda_1^3+\lambda_0^3\lambda_2^3+\lambda_1^3\lambda_2^3][(\mathrm{e}^{2K}+2\,\mathrm{e}^{-K})^M-(\mathrm{e}^{2K}-\mathrm{e}^{-K})^M].$$

Here the λ_j^3 are functions of $\beta\exp(-K)$ and using the values of $\lambda_j^3(0)$ for $j=0,1,2$, we obtain then the partition function:

$$\mathscr{Z}_M=(2\,\mathrm{e}^{-K}+\mathrm{e}^{2K})^M+2(\mathrm{e}^{2K}-\mathrm{e}^{-K})^M.$$

This has also been discussed in Kwasniewski (1987).

For arbitrary N a similar calculation leads to the following simple result:

$$\mathscr{Z}_M=[(N-1)\,\mathrm{e}^{-K}+\mathrm{e}^{(N-1)K}]^M+(N-1)(\mathrm{e}^{(N-1)K}-\mathrm{e}^{-K})^M.$$

This result can also be checked using an alternative method of computing the trace of cyclic matrices (Audit 1985)

$$\mathrm{Trace}(\mathbf{T})^M=\sum_{l=1}^{N-1}\left(\mathrm{e}^{(N-1)K}+\mathrm{e}^{-K}\sum_{k=1}^{N-1}(\omega)^{kl}\right)$$
$$=[\mathrm{e}^{(N-1)K}+(N-1)\,\mathrm{e}^{-K}]^M+(N-1)[\mathrm{e}^{(N-1)K}-\mathrm{e}^{-K}]^M.$$

5. Conclusions

In this paper we have shown how a class of special functions introduced not long ago may turn out to be very useful in mathematical physics. The properties of these functions lend themselves to the construction of advantageous formulation of otherwise algebraically complicated problems. We have illustrated the use of these functions in some special instances of statistical mechanics and show how to work with them. It is hoped that many more circumstances where they turn out to be of much more convenient use, will occur.

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